SINGULAR-UNBOUNDED RANDOM JACOBI MATRICES

NISHANT RANGAMANI

Abstract. There have been several recent proofs of one-dimensional Anderson localization based on positive Lyapunov exponent that hold for bounded potentials. We provide a Lyapunov exponent based proof for unbounded potentials, simultaneously treating the singular and unbounded Jacobi case by extending the techniques in [19].

1. Introduction

In this paper, we consider random Jacobi operators given by:

$$H_\omega\psi(n) = t_\omega(n-1)\psi(n-1) + t_\omega(n)\psi(n+1) + V_\omega(n)\psi(n),$$

where $\{V_\omega(n)\}_{n=-\infty}^{\infty}$ and $\{t_\omega(n)\}_{n=-\infty}^{\infty}$ are two i.i.d processes, independent of each other on some probability space Ω.

When the random variable $t_\omega(n) = 1$ almost surely, this is the celebrated Anderson model in one dimension. Many proofs exist for regular distribution of the potential (see [1–3, 11, 12, 22, 25, 26]). The first proof of spectral localization for non-constant potentials requiring only a finite moment and no additional regularity on the distribution was given by Carmona, Klein, and Martinelli [6] in 1987 and represented a significant advancement, resolving the issue of localization for Bernoulli disorder. Their proof is based on multi-scale analysis. Recently, there have been several new proofs in dimension one (see [5, 15, 19]) which take advantage of positive Lyapunov exponents. The proof given in [19] provides a very short argument by using some ideas Jitomirskaya introduced in 1999 in the study of the almost Mathieu operator [20]. The arguments in [5, 15, 19] all hold for bounded potentials. Here, we extend the argument of [19] to the unbounded Schrödinger case, as well as the unbounded and singular Jacobi case. We believe the unbounded and singular Jacobi results are new although they likely could have been obtained by multi-scale analysis as well. Our approach, however, is significantly simpler.

We now elaborate on the precise set-up of the operators described above. Let Ω_0 be $\mathbb{R}^+ \times \mathbb{R}$ with probability measure μ_1 on $\mathbb{R}^+$, μ_2 on $\mathbb{R}$ and $\mu := \mu_1 \times \mu_2$ on Ω_0. Then, with $\Omega = \Omega_0^{\mathbb{Z}}$, $\mathbb{P} = \mu^{\mathbb{Z}}$, and $\omega(n) = (\omega_1(n), \omega_2(n))$, we let $t_\omega(n) = \omega_1(n)$ and $V_\omega(n) = \omega_2(n)$. Additionally, we have the associated shift operator on Ω given by $T(\omega(n)) = \omega(n-1)$, which is ergodic. Ergodicity provides the foundation for the study of such random models. In particular, under these general conditions, Kirsch and Martinelli have shown that almost surely, the spectrum $\sigma(H_\omega)$ is a non-random set [21].

We shall further suppose that $V_\omega(0)$ is *a.s.* non-constant. The only other conditions will be given by finiteness of certain moments of these processes. In particular, we require $\mathbb{E}[|V_\omega(0)|^\alpha] < \infty$, $\mathbb{E}[(1/t_\omega(0))^\beta] < \infty$, and $\mathbb{E}[(t_\omega(0))^\nu] < \infty$, for some α, β, and $\nu > 0$. Since H_ω is self-adjoint if $\sum_{n=-\infty}^{\infty} \frac{1}{t_\omega(n)} = \infty$ (e.g. Lemma 2.16 in [27]), by the law of large numbers, H_ω is self-adjoint for a.e. ω. Thus, we will obtain localization in the following three specific contexts of interest:

i) $V_\omega(0)$ is unbounded and $t_\omega(0) = 1$ a.s.
This is simply the Anderson model with an unbounded potential, recovering the result of Carmona, Klein, and Martinelli [6].
ii) $V_\omega(0)$ is bounded (but not a.s. constant), while $t_\omega(0)$ is unbounded and/or singular
i.e. $t_\omega(0) \in (0, \infty)$ a.s. (as opposed to $t_\omega(0) \in [M_1, M_2]$ a.s. with $0 < M_1 \leq M_2$)
iii) $V_\omega(0)$ is unbounded, $t_\omega(0)$ is unbounded and/or singular in the same sense as case ii).

In the quasi-periodic case, these singular and unbounded operators arise naturally. The extended Harper's model, proposed by Thouless and studied in [4, 17, 18, 29] provides an example of the singular case, and the Maryland model (see [9, 10, 24]) provides an example of the unbounded case.

Finally, we mention that to the best of our knowledge, ii) and iii) have not been previously obtained by other methods (without further regularity assumptions).

The rest of the paper is organized as follows: Section 2 contains the statement of the main result (Theorem 1), Section 3 gathers the various formulas and miscellaneous objects needed. Section 4 describes the application of results used in [19] when $t_\omega(0)$ and $V_\omega(0)$ are unbounded. Section 5 contains a reformulation of Theorem 1 as Theorem 10, Section 6 contains the lemmas which constitute the central revisions needed to adapt the argument in [19] in the unbounded and/or singular case, and finally, Section 7 contains the proof of Theorem 10. We note that given the adjustments made in Sections 4 through 6, the proof of Theorem 10 follows the argument in [19] closely.

2. Preliminaries and Statement of the Main Theorem

Definition 1. The operators H_ω are said to exhibit *spectral localization* if for *a.e* ω, H_ω has pure point spectrum and all of its eigenfunctions decay exponentially in n.

Definition 2. We call E a *generalized eigenvalue* (*g.e.*), if there exists a nonzero polynomially bounded function $\psi(n)$ such that $H_\omega(\psi) = E\psi$. We call $\psi(n)$ a *generalized eigenfunction.*

Since the spectral measures are supported by the set of generalized eigenvalues (e.g. [16]), spectral localization is a consequence of:

Theorem 1. *For a.e. ω, for every g.e. E, the corresponding generalized eigenfunction $\psi_{\omega,E}(n)$ decays exponentially in n.*

3. Terminology and Formulas

We let $H_{\omega,[a,b]}$ denote the operator H_ω restricted to the interval $[a, b]$ with zero boundary condition, let $\sigma(H_{\omega,[a,b]})$ denote its spectrum, and for $j \in [1, b - a + 1]$, let $E_{j,[a,b],\omega}$ be the eigenvalues of $H_{\omega,[a,b]}$.

For a discrete Jacobi operator H_ω, we denote the *Green's function* on the interval $[a,b]$ with energy $E \notin \sigma(H_{\omega,[a,b]})$ and zero boundary condition as

$$G_{[a,b],E,\omega} = (H_{\omega,[a,b]} - E)^{-1} \tag{1}$$

$G_{[a,b],E,\omega}$ can be viewed as a $b-a+1$-dimensional matrix and we denote the x, y entry of this matrix as $G_{[a,b],E,\omega}(x,y)$.

We also let $P_{[a,b],E,\omega} = \det(H_{\omega,[a,b]} - E)$ and $\mathbb{P}_{[a,b]}$ be $\mu^{[a,b]\cap\mathbb{Z}}$ on $\Omega_0^{[a,b]\cap\mathbb{Z}}$.

Definition 3. For $c > 0$ and $n \in \mathbb{N}$, we say $x \in \mathbb{Z}$ is (c,n,E,ω)-*regular* if

(1) $|G_{[x-n,x+n],E,\omega}(x,x-n)| \leq \dfrac{1}{t_\omega(x)} e^{-cn}$,

(2) $|G_{[x-n,x+n],E,\omega}(x,x+n)| \leq \dfrac{1}{t_\omega(x+n)} e^{-cn}$

Definition 4. We say $x \in \mathbb{Z}$ is (c,n,E,ω)-*singular* if it is not (c,n,E,ω)-*regular.*

We first discuss some consequences of ergodicity, then set some conventions and list some additional objects and formulas below.

(1) We let

$$T_{k,E,\omega} := \begin{pmatrix} \frac{E - V_\omega(k)}{t_\omega(k)} & \frac{-1}{t_\omega(k)} \\ t_\omega(k) & 0 \end{pmatrix},$$

so that for a generalized eigenfunction ψ_ω of H_ω, we have:

(2)

$$T_{k,E,\omega,} \begin{pmatrix} \psi_\omega(k) \\ \psi_\omega(k-1)t_\omega(k-1) \end{pmatrix} = \begin{pmatrix} \psi_\omega(k+1) \\ \psi_\omega(k)t_\omega(k) \end{pmatrix}.$$

(3) Moreover, for any interval $[a,b]$, we set $S_{[a,b],E,\omega} = \prod_{k=b}^{a} T_{k,E,\omega}$, so that

(4) $S_{[a,b],E,\omega} \begin{pmatrix} \psi_\omega(a) \\ \psi_\omega(a)t_\omega(a-1) \end{pmatrix} = \begin{pmatrix} \psi_\omega(b+1) \\ \psi_\omega(b)t_\omega(b) \end{pmatrix}.$

Since the shift operator T on Ω is ergodic and $\mathbb{E}[\ln^+ ||T_{k,E,\omega}||] < \infty$, we can apply Kingman's subadditive ergodic theorem to obtain the Lyapunov exponent $\gamma(E) \overset{a.e.\ \omega}{:=} \lim_{n\to\infty} \dfrac{\ln ||S_{[1,n],E,\omega}||}{n}$.

Remark. Note that the above limit exists *a.e.* ω for a fixed energy E.

We now list Jacobi analogs of some well-known formulas from the Schrödinger case (the proofs in both cases are identical).

(5) For a generalized eigenfunction ψ of H_ω, and $x \in [a,b]$
$\psi(x) = -G_{[a,b],E,\omega}(x,a)\psi(a-1)t_\omega(a-1) - G_{[a,b],E,\omega}(x,b)\psi(b+1)t_\omega(b)$.

(6) For $x \leq y$, we have:

$$|G_{[a,b],E,\omega}(x,y)| = \frac{|P_{[a,x-1],E,\omega}| t_\omega(x) \cdots t_\omega(y-1) |P_{[y+1,b],E,\omega}|}{|P_{[a,b],E,\omega}|},$$

where
$P_{[a,b],E,\omega} := 1$, if $b < a$.

$t_\omega(a)\cdots t_\omega(b) := 1$, if $b < a$.
$t_\omega(a)\cdots t_\omega(b) := t_\omega(a)$, if $a = b$.

This formula connecting the Green's function to determinants of restricted H_ω's can be proven by an application of Cramer's rule.

$$S_{[a,b],E,\omega} = \begin{pmatrix} \frac{P_{[a,b],E,\omega}}{t_\omega(a)\cdots t_\omega(b)} & \frac{-P_{[a+1,b],E,\omega}}{t_\omega(a)\cdots t_\omega(b)} \\ \frac{P_{[a,b-1]}(t_\omega(b))^2}{t_\omega(a)\cdots t_\omega(b)} & \frac{-P_{[a+1,b-1]}(t_\omega(b))^2}{t_\omega(a)\cdots t_\omega(b)} \end{pmatrix}. \tag{7}$$

This last formula expressing the transfer matrices through determinants of restricted H_ω's can be established by induction.

Since we aim to prove localization at energies in a finite interval $\tilde{I}$, we fix $\tilde{I} = [s,t]$, a compact interval with non-empty interior in $\mathbb{R}$ and let $I = [s-1, t+1]$. We then define the following 'large deviation' sets:

$$B^+{}_{[a,b],\varepsilon} = \left\{(E,\omega) : E \in I, \frac{|P_{[a,b],E,\omega}|}{t_\omega(a)\cdots t_\omega(b)} \geq e^{(\gamma(E)+\varepsilon)(b-a+1)}\right\}, \tag{8}$$

$$B^-{}_{[a,b],\varepsilon} = \left\{(E,\omega) : E \in I, \frac{|P_{[a,b],E,\omega}|}{t_\omega(a)\cdots t_\omega(b)} \leq e^{(\gamma(E)-\varepsilon)(b-a+1)}\right\}, \tag{9}$$

and denote the corresponding sections by

$$B^\pm{}_{[a,b],\varepsilon,\omega} = \left\{E : (E,\omega) \in B^\pm_{[a,b],\varepsilon}\right\}, \tag{10}$$

$$B^\pm{}_{[a,b],\varepsilon,E} = \left\{\omega : (E,\omega) \in B^\pm_{[a,b],\varepsilon}\right\}. \tag{11}$$

Additionally, we let

$$B_{[a,b],*} = B^+_{[a,b],*} \cup B^-_{[a,b],*} \tag{12}$$

and finally,

$$\tilde{B^+}{}_{[a,b],\varepsilon} = \left\{(E,\omega) : \frac{|P_{[a,b],E,\omega}|}{t_\omega(a)\cdots t_\omega(b)} \geq e^{(\gamma(E)+\varepsilon)(b-a+1)}\right\}, \tag{13}$$

$$\tilde{B^-}{}_{[a,b],\varepsilon} = \left\{(E,\omega) : \frac{|P_{[a,b],E,\omega}|}{t_\omega(a)\cdots t_\omega(b)} \leq e^{(\gamma(E)-\varepsilon)(b-a+1)}\right\}, \tag{14}$$

with the corresponding extensions to the sections as in (10) and (11) above.

By rescaling the operator in question, we may assume that $\mathbb{E}[(\frac{1}{t_\omega(0)})^\beta] = c_1 < 1$. Furthermore, we let $\mathbb{E}[t_\omega(0)^\nu] = c_2$ and $\mathbb{E}[|V_\omega(0))|^\alpha] = c_3$.
Lastly, we let $d = \inf\{\gamma(E) : E \in I\}$ and note here that $d > 0$. For the proof, see Theorem 7.

4. Accounting for Unboundedness/Singularity

There are a variety of results that the argument given in [19] relies on, the most important results being positivity of the Lyapunov exponent, an estimate by Craig-Simon [7] that provides uniform bounds on transfer matrices, and certain large deviation estimates [28]. That these theorems can be applied to bounded random Schrödinger operators is understood and it is our aim in this section to describe their extension to the singular-unbounded Jacobi case.

We begin by describing the setting of the theorem by Craig-Simon.

Suppose $M_\omega = \Delta + V_\omega$ is a one-dimensional discrete random Schrödinger operator where the potential V is a bounded ergodic process, with

$$A_{k,E,\omega} := \begin{pmatrix} E - V_\omega(k) & -1 \\ 1 & 0 \end{pmatrix},$$

and $R_{[a,b],E,\omega} := \prod_{k=b}^{a} A_{k,E,\omega}$. Additionally, let $\overline{\gamma}^+(\omega, E) = \limsup_{n\to\infty} \frac{\ln ||R_{[1,n],E,\omega}||}{|n|}$, $\overline{\gamma}^-(\omega, E) = \limsup_{n\to\infty} \frac{\ln ||R^{-1}_{[-n,-1],E,\omega}||}{|n|}$ and $\gamma(E)$ denote the associated Lyapunov exponent.

Theorem 2. *(Craig-Simon [7]) In the above setting, for a.e. ω and all E, $\overline{\gamma}^{\pm}(\omega, E) \leq \gamma(E)$.*

The proof given in [7] only requires that the random process through which the operator is defined is ergodic, in addition to the finiteness of $\mathbb{E}[\ln^+ ||A_{k,E,\omega}||]$. Both these requirements hold for H_ω, given that the diagonal and off-diagonal elements are i.i.d. and our assumptions on the expectations of $t_\omega(0)$, $\frac{1}{t_\omega(0)}$,and $V_\omega(0)$. Specifically, the finiteness of the above quantity is used to ensure the application of Kingman's subadditive ergodic theorem which not only results in the almost sure existence of the Lyapunov exponent for each energy E, but also that $\overline{\gamma}^{\pm}(\omega, E)$ is submean and $\gamma(E)$ is subharmonic. It is this last fact that is proved in the Craig-Simon paper and is then used to give a proof of the theorem as stated.

With $T_{k,E,\omega}$ and $S_{[a,b],E,\omega}$ defined as in Section 3, $\overline{\gamma}^+(\omega, E) = \limsup_{n\to\infty} \frac{\ln ||S_{[1,n],E,\omega}||}{|n|}$, $\overline{\gamma}^-(\omega, E) = \limsup_{n\to\infty} \frac{\ln ||S^{-1}_{[-n,1],E,\omega}||}{|n|}$, and $\gamma(E)$ the Lyapunov exponent, we have:

Theorem 3. *(Unbounded Craig-Simon [7]) For a.e. ω and all E, $\overline{\gamma}^{\pm}(\omega, E) \leq \gamma(E)$.*

This result's primary role in our argument is through the following restatement and subsequent corollary.

Theorem 4. *For a.e. ω, for all E, we have*

$$\max\left\{\limsup_{n\to\infty} \frac{\ln ||S^{-1}_{[-n,-1],E,\omega}||}{n}, \limsup_{n\to\infty} \frac{\ln ||S_{[1,n],E,\omega}||}{n}\right\} \leq \gamma(E), \tag{2}$$

$$\max\left\{\limsup_{n\to\infty} \frac{\ln ||S_{[n+1,2n],E,\omega}||}{n}, \limsup_{n\to\infty} \frac{\ln ||S_{[2n+2,3n+1],E,\omega}||}{n}\right\} \leq \gamma(E). \tag{3}$$

Remark. The first statement in Theorem 4 is an immediate consequence of Theorem 3 and the second statement can be obtained by the same proof.

Corollary 1. *For a.e. ω, for every E and any $\epsilon > 0$, there is $N_2(\omega, E, \epsilon)$ such that for every $n > N_2$ we have*

$$\max\{||S^{-1}_{[-n,-1],E,\omega}||, ||S_{[1,n,E,\omega}||\} < e^{(\gamma(E)+\epsilon)(n)}. \tag{4}$$

$$\max\{||S_{[n+1,2n],E,\omega}||, ||S_{[2n+2,3n+1],E,\omega}||\} < e^{(\gamma(E)+\epsilon)(n)}. \tag{5}$$

We now turn to the positivity of the Lyapunov exponent.

As above, we describe the setting of the theorem first. Suppose $\{t_\omega(n)\}$ and $\{V_\omega(n)\}$ are two i.i.d. processes independent of each other with $V_\omega(0)$ a.s. non-constant, for some $c > 0$, $t_\omega(0) > c$ a.s., and $\mathbb{E}[ln(1 + t_\omega(0) + |V_\omega(0)|)] < \infty$.

Theorem 5. *(Figotin [23]) Suppose the H_ω's are random Jacobi operators defined through the processes $\{t_\omega(n)\}_{n=-\infty}^{\infty}$ and $\{V_\omega(n)\}_{n=-\infty}^{\infty}$, satisfying the above conditions, then the corresponding Lyapunov exponent $\gamma(E)$ is strictly positive for any $E \in \mathbb{R}$.*

We first note that condition on the expectation in the above theorem is required to ensure that $\mathbb{E}[\ln^+ ||T_{k,E,\omega}||] < \infty$. While they suppose $t_\omega(0)$ is bounded from below, this is only needed to ensure $\mathbb{E}[\ln^+ ||T_{k,E,\omega}||] < \infty$. As such, their result applies in our setting since we have a condition on the expectation of $\frac{1}{t_\omega(0)}$ which implies the finiteness of $\mathbb{E}[\ln^+ ||T_{k,E,\omega}||])$ as well.

Moreover, we remark that the argument of [23] proceeds by showing that the conditions from a theorem by Fürstenberg [13], which guarantees positivity of the Lyapunov exponent, hold. In particular, they show that the transfer matrices $(T_{k,E,\omega})$ with common distribution in $SL(2, \mathbb{R})$ denoted by ρ and the smallest closed subgroup containing the support of ρ denoted by G_ρ satisfy

ii) G_ρ is not compact.

iii) There is no non-trivial G_ρ invariant probability measure on $\mathbb{RP}^1$.

These two facts will become relevant in showing not only that $d > 0$ (see Theorem 6 and 7 below), but also that the result on large deviations of matrix elements [28] applies in our setting.

Theorem 6. *$\gamma(E)$ is continuous on $\mathbb{R}$.*

Proof. Fix $E \in \mathbb{R}$ and let E_k be a sequence in $\mathbb{R}$ such that $E_k \to E$ as $k \to \infty$. Let μ_E denote the probability measure on $SL(2, \mathbb{R})$ obtained through the transfer matrices $T_{n,E,\omega}$ and let G_{μ_E} denote the smallest closed subgroup containing the support of μ_E. Note that by iii) above, there is no non-trivial subspace $W \subset \mathbb{R}^2$ such that W is G_{μ_E}-invariant. Thus, the hypothesis of Theorem B in [14] holds (i.e. that there can be at most one such W).

Now let $X_k : \Omega \to SL(2, \mathbb{R})$ be defined by $X_k(\omega) = T_{0,E_k,\omega}$ and $X : \Omega \to SL(2, \mathbb{R})$ be defined by $X(\omega) = T_{0,E,\omega}$. By Theorem B in [14], to prove $\gamma(E_k) \to \gamma(E)$ as $k \to \infty$, it suffices to show:

(1) For any $h : SL(2, \mathbb{R}) \to \mathbb{C}$ with h continuous and of compact support, $\mathbb{E}[h(X_k)] \to \mathbb{E}[h(X)]$ as $k \to \infty$
(2) $\mathbb{E}[\log^+(||X_k||\chi_{\{||X_k|| \geq n\}})] \to 0$ as $n \to \infty$ uniformly in k
(3) $\mathbb{E}[\log^+(||X^{-1}||\chi_{\{||X^{-1}|| \geq n\}})] \to 0$ as $n \to \infty$.

Remark. Condition (1) above is known as *weak convergence* and conditions (2) and (3) together are known as *bounded convergence*.

Returning to the proof, (1) follows by dominated convergence since $X_k \to X$ for a.e. ω and h is continuous and of compact support. Now choose $M \in \mathbb{R}$ so that

$|E_k| \leq M$ for all k and let $\eta = \min\{\alpha, \beta, \nu\}$. Note we have $||X_k(\omega)|| \leq Y(\omega) = \sqrt{2}\max\{\frac{M+|V_\omega(0)|}{t_\omega(0)}, \frac{1}{t_\omega(0)}, t_\omega(0)\}$. Thus, $\eta \log^+ ||X_k|| \leq Y^\eta$ and we have $\mathbb{E}[Y^\eta] < \infty$ by our hypotheses. It follows that $\{\log^+ ||X_k||\}$ is a uniformly integrable family, so (2) holds. Finally, (3) follows since $\mathbb{E}[\log^+ ||X^{-1}||] < \infty$. We conclude that $\gamma(E_0)$ is a continuous function on $\mathbb{R}$.

□

Theorem 7. *For d defined as in Section 3, (i.e. $d = \inf\{\gamma(E) : E \in I\}$), we have $d > 0$.*

Proof. By Theorem 5 (in particular, the discussion following the theorem), we have $\gamma(E) > 0$ for all $E \in \mathbb{R}$. The result now follows by Theorem 6 and compactness of I.

□

We now deal with the aforementioned large deviation result.

Again, we begin with the setting of the theorem.

Suppose $\{Y_k\}$ are i.i.d. 2×2 matrices with common distribution μ, where μ is a probability measure on $SL(2, \mathbb{R})$. Let $l(M) = \max\{ln^+||M||, ln^+||M^{-1}||\}$ and suppose for some $\tau > 0, \int \exp(\tau l(M))d(\mu(M)) < \infty$. Moreover, suppose G_μ, the smallest subgroup containing the support of μ, is both strongly irreducible and contracting.

Theorem 8. *(Tsay [28]) In the above setting, for each $\epsilon > 0$, there is an $a > 0$ such that for all unit vectors $u, v \in \mathbb{R}^2$, $\mathbb{P}\{|\frac{1}{n}log|\langle\prod_{k=1}^{n} Y_k u, v\rangle| \geq \epsilon\} \leq e^{-an}$ for sufficiently large n.*

Tsay goes on to extend this result when the matrices (and distributions) depend on a real parameter E in a fixed compact set F. That is, suppose $Y_{k,E}$ are i.i.d. 2×2 matrices in $SL(2, \mathbb{R})$ with respective probability measures μ_E. If there is $C > 0$ and $\tau > 0$ such that $\int exp(\tau l(M))d(\mu_E(M)) < C$ for all $E \in F$ (in addition to $Y_{k,E}$ satisfying the conditions in Theorem 8), then Theorem 8 holds uniformly in E.

Theorem 9. *(Tsay [28]) In the above setting, for each $\epsilon > 0$, there is an $a > 0$ such that for all unit vectors $u, v \in \mathbb{R}^2$, $\mathbb{P}\{|\frac{1}{n}log|\langle\prod_{k=1}^{n} Y_{k,E}u, v\rangle| \geq \epsilon\} \leq e^{-an}$ for sufficiently large n uniformly in E.*

We call the a in the above theorem the 'large deviation parameter' associated with ϵ.

We now show that the conditions of Theorems 8 and 9 hold in our setting. We begin by explaining the terminology used in the statement. The term *strongly irreducible* means that there is no finite union of proper subspaces $W \subset \mathbb{R}^2$ such that $M(W) = W$ for all $M \in G_\mu$. The term *contracting* means that there is a sequence in G_μ say $\{M_n\}$ such that $\frac{M_n}{||M_n||}$ converges to a rank one matrix. Firstly, by taking M_n in G_μ with $||M_n|| \to \infty$ and considering a convergent subsequence of $\frac{M_n}{||M_n||}$, it follows that condition ii) (non-compactness of G_μ) implies contracting. Next, if strong irreducibility does not hold, this implies the existence of a non-empty, finite $L \subset \mathbb{RP}^1$ such that $M(L) = L$ for all $M \in G_\mu$. Indeed, taking the sum of point masses with weight $\frac{1}{|L|}$ at each of the points of L gives a non-trivial G_μ invariant probability measure on $\mathbb{RP}^1$ and we conclude iii) implies strong irreducibility. Finally, the required moment condition is easily seen to be satisfied given our assumptions on the various moments of $V_\omega(0), t_\omega(0)$,and $\dfrac{1}{t_\omega(0)}$ and compactness of I.

Theorem 9 finds its use in our argument through the following corollary:

Corollary 2. *(Large Deviations [28]) For any $\varepsilon > 0$, there is $\eta > 0$ and $N_0 \in \mathbb{N}$ such that for $b - a > N_0$,* $\mathbb{P}\left\{\omega : \left|\left(\frac{1}{b-a+1}\log\frac{|P_{[a,b],E,\omega}|}{t_\omega(a)\cdots t_\omega(b)}\right) - \gamma(E)\right| \geq \varepsilon\right\} \leq e^{-\eta(b-a+1)}$.

Proof. The transfer matrices $T_{k,E,\omega}$ are certainly i.i.d. Additionally, by the above discussion, they also satisfy the irreducibility, contracting, and expectation condition. The corollary now follows by taking $u = (1,0)$, $v = (1,0)$ and applying formula (7). □

5. Main Theorem Reformulated

Given the presence of $\frac{1}{t_\omega(k)}$'s in the definition of regularity, we first require a lemma before presenting the reduction of Theorem 1 to Theorem 10.

Let $\lambda = \min\{\beta, \nu\}$.

Lemma 1. *If $r > 1$ and* $A_n = \left\{\omega : \frac{t_\omega(n)}{t_\omega(2n+1)} \geq n^{\frac{r}{\lambda}} \text{ or } \frac{t_\omega(n)}{t_\omega(2(n+1))} \geq n^{\frac{r}{\lambda}}\right\}$, *then* $\mathbb{P}[A_n \text{ i.o.}] = 0$.

Proof. Put $J_n = \left\{\omega : \frac{t_\omega(n)}{t_\omega(2n+1)} \geq n^{\frac{r}{\lambda}}\right\}$ and $K_n = \left\{\omega : \frac{t_\omega(n)}{t_\omega(2(n+1))} \geq n^{\frac{r}{\lambda}}\right\}$. Then we have:

$$n^r\mathbb{P}[J_n] \leq \mathbb{E}[(t_\omega(n))^\lambda]\mathbb{E}\left[\frac{1}{(t_\omega(2n+1))^\lambda}\right] \leq \mathbb{E}[(t_\omega(n))^\nu]\mathbb{E}\left[\frac{1}{(t_\omega(2n+1))^\beta}\right] = c_1c_2.$$

The first inequality is by Chebyshev, together with independence, and the second inequality follows since $\lambda = \min\{\nu, \beta\}$. Applying the same argument to K_n, we have $n^r\mathbb{P}[K_n] \leq c_1c_2$. Since A_n is the union of K_n and J_n, $\mathbb{P}[A_n] \leq \dfrac{2c_1c_2}{n^r}$ and the result follows by Borel-Cantelli. □

Thus, with the above lemma and formula (5), Theorem 1 follows from:

Theorem 10. *For a.e. ω, for every generalized eigenvalue E of H_ω, there is $C(E) > 0$ and $N(\omega, E)$ such that for $n > N$, $2n$ and $2n+1$ are $(C(E), n, E, \omega)$-regular.*

6. Lemmas

Lemma 2. *Let $n \geq 2$ and suppose $0 < 8\epsilon_0 < d$. If x is $(\gamma(E) - 8\varepsilon_0, n, E, \omega)$-singular, then* $(E,\omega) \in {B^-}_{[x-n,x+n],\epsilon_0} \cup {B^+}_{[x-n,x-1],\epsilon_0} \cup {B^+}_{[x+1,x+n],\epsilon_0}$.

Proof. This follows by the definition of $(\gamma(E) - 8\varepsilon_0, n, E, \omega)$- *singularity* and formula (6). □

Remark. Lemmas 3 and 4 follow [19] very closely with some minor modifications needed to deal with unbounded and/or singular $\{t_\omega(n)\}$ and unbounded $\{V_\omega(n)\}$.

Let m denote Lebesgue measure on $\mathbb{R}$.

Lemma 3. *Suppose* $0 < \epsilon_0 < d$, η_0 *is the corresponding large deviation parameter (from Corollary 2), and* $0 < \delta_0 < \eta_0$, *then for a.e.* ω, *there is* $N_1(\omega)$ *such that for* $n > N_1$, $\max\left\{m(B^-_{[n+1,3n+1],\epsilon_0,\omega}), m(B^-_{[-n,n],\epsilon_0,\omega})\right\} \le e^{-(\eta_0-\delta_0)(2n+1)}$.

Proof. We have

$$\begin{aligned} m \times \mathbb{P}\left(B^-_{[a,b],\varepsilon_0}\right) &= \mathbb{E}\left(m\left(B^-_{[a,b],\varepsilon_0,\omega}\right)\right) \\ &= \int_{\mathbb{R}} \mathbb{P}\left(B^-_{[a,b],\varepsilon_0,E}\right)\, dm(E) \\ &\le m(I)e^{-\eta_0(b-a+1)} \end{aligned}$$

The first two equalities are simply Fubini's theorem, and the inequality follows by Corollary 2.

Let

$$F_n = \left\{\omega : m\left(B^-_{[n+1,3n+1],\epsilon_0,\omega}\right) \ge e^{-(\eta_0-\delta_0)(2n+1)}\right\},$$

and

$$G_n = \left\{\omega : m\left(B^-_{[-n,n],\epsilon_0,\omega}\right) \ge e^{-(\eta_0-\delta_0)(2n+1)}\right\}.$$

We have, $e^{-(\eta_0-\delta_0)(2n+1)}\mathbb{P}(F_n) \le \mathbb{E}\left(m\left(B^-_{[n+1,3n+1],\varepsilon_0,\omega}\right)\right) \le m(I)e^{-\eta_0(2n+1)}$, with a similar estimate holding for G_n. The first inequality is Chebyshev and the second follows by the first line of the proof.

Thus,

$$\mathbb{P}(F_n \cup G_n) \le 2m(I)e^{-\delta_0(2n+1)}$$

and the result follows by Borel-Cantelli. □

Lemma 4. *Suppose* $0 < \epsilon < d$, $\eta_\epsilon > 0$ *is the corresponding large deviation parameter and* $p > \dfrac{6}{\eta_\epsilon}$. *For* $n \in \mathbb{N}$, *put* $C_n = \{\omega : \exists y \in [-n,n], |-n-y| \ge \ln(n^p)$, *and* $E_{j,[n+1,3n+1],\omega} \in B_{[-n,y],\varepsilon,\omega}$ *for some* $1 \le j \le 2n+1\}$ *and* $D_n = \{\omega : \exists y \in [-n,n], |n-y| \ge \ln(n^p), E_{j,[n+1,3n+1],\omega} \in B_{[y,n],\varepsilon,\omega}$ *for some* $1 \le j \le 2n+1\}$. *Then* $\mathbb{P}[C_n \cup D_n \ i.o.] = 0$.

Proof. Fix $n \in \mathbb{N}$, y with $|-n-y| \ge \ln(n^p)$, and $1 \le j \le 2n+1$, and put $A_{n,y,j} = \{\omega : E_{j,[n+1,3n+1],\omega} \in B_{[-n,y]}\}$.

Since $[n+1, 3n+1] \cap [-n, n] = \emptyset$, by independence and Corollary 2 we have

$$\mathbb{P}\left(B_{[-n,y],\epsilon,E_{j,[n+1,3n+1],\omega}}\right) = \mathbb{P}_{[-n,y]}\left(B_{[-n,y],\epsilon,E_{j,[n+1,3n+1],\omega}}\right) \le e^{-\eta_\epsilon|-n-y|}.$$

Indeed, for each n, if $Q_n' = \{y \in [-n,n] : |-n-y| \ge \ln(n^p)\}$,

$$C_n = \bigcup_{y\in Q_n',\ 1\le j\le 2n+1} A_{n,y,j}.$$

By the above, we have $\mathbb{P}[C_n] \le (2n+1)^2 e^{-\eta_\epsilon \ln(n^p)}$. Thus $\mathbb{P}[C_n \ \ i.o] = 0$ by Borel-Cantelli. The result follows by applying the same argument to D_n. □

Lemma 5. *Suppose $p > 0$ and $r > 1$. Let $J_n = \{\omega : \exists k \in [-n, n], |-n-k| \leq \ln(n^p)$ or $|k-n| \leq \ln(n^p)$ and $|V_\omega(k)| \geq n^{\frac{r}{\alpha}}$ or $|t_\omega(k)| \geq n^{\frac{r}{\nu}}\}$. Then $\mathbb{P}[J_n \ i.o.] = 0$.*

Proof. Put $Q_n = \{k \in [-n, n] : |-n-k| \leq \ln(n^p)$ or $|n-k| \leq \ln(n^p)\}$, and $A_k = \{\omega : |V_\omega(k)| \geq n^{\frac{r}{\alpha}}$ or $t_\omega(k) \geq n^{\frac{r}{\nu}}\}$. Then $J_n = \bigcup_{k \in Q_n} A_k$.

By Chebyshev and stationarity, for any $k \in Q_n$, $\mathbb{P}[A_k] \leq \dfrac{c_2 + c_3}{n^r}$. Thus, $\mathbb{P}(J_n) \leq 2(c_2 + c_3)(\ln(n^p) + 1)n^{-r}$. By Borel-Cantelli, $\mathbb{P}[J_n \ i.o.] = 0$. □

Let $q = \max\left(\frac{r}{\alpha}, \frac{r}{\nu}\right)$.

Corollary 3. *If $p > 0$ and $r > 1$, for a.e. ω, there is $N(\omega)$ such that for $n > N$ and any $k \in [-n, n]$ st. $|-n-k| \leq \ln(n^p)$ (respectively, $|n-k| \leq \ln(n^p)$),*

$$|P_{[-n,k],\omega}| \leq n^{q(\ln(n^p)+1)} \quad (\textit{respectively,} \quad |P_{[k,n],\omega}| \leq n^{q(\ln(n^p)+1)}).$$

Remark. Proceeding in the same manner as in the above lemma, we can obtain the following three lemmas. We prove Lemmas 6 and 7 and exclude the proof of Lemma 8, as its proof is identical to the proof of Lemma 7.

Lemma 6. *Suppose $r > 2$. If $A_n = \left\{\omega : \exists k \in [-n, n] \text{ s.t. } \dfrac{1}{t_\omega(k)} > n^{\frac{r}{\beta}}\right\}$, then $\mathbb{P}[A_n \ i.o.] = 0$.*

Proof. Put $J_{k,n} = \left\{\omega : \dfrac{1}{t_\omega(k)} \geq n^{\frac{r}{\beta}}\right\}$, then by Chebyshev, $n^r \mathbb{P}[J_{k,n}] \leq \mathbb{E}[\frac{1}{(t_\omega(k))^\beta}] = c_1$ and this holds for all $n \in \mathbb{N}$ and $k \in [-n, n]$ because the process $\{t_\omega(m)\}_{m=-\infty}^{\infty}$ is stationary. Since $A_n = \bigcup_{k \in [-n,n]} J_{k,n}$, we obtain, $\mathbb{P}[A_n] \leq \dfrac{c_1(2n+1)}{n^r}$. The result now follows by Borel-Cantelli. □

Lemma 7. *Suppose $r > 1$. If*

$$A_n = \left\{\omega : \exists k \in [-n, n], |-n-k| \leq \ln(n^p) \text{ and } \frac{1}{t_\omega(-n) \cdots t_\omega(k-1)} \geq n^{\frac{r}{\beta}}\right\},$$

then $\mathbb{P}[A_n \ i.o.] = 0$.

Proof. Put $Q_n' = \{k \in [-n, n] : |-n-k| \leq \ln(n^p)\}$, and

$$J_{k,n} = \left\{\omega : \frac{1}{t_\omega(-n) \cdots t_\omega(k-1)} \geq n^{\frac{r}{\beta}}\right\}.$$

Then $A_n = \bigcup_{k \in Q_n'} J_{k,n}$.

Hence, $n^r \mathbb{P}[J_{k,n}] \leq \mathbb{E}\left[\dfrac{1}{(t_\omega(-n) \cdots t_\omega(k-1))^\beta}\right] = (c_1)^{|-n-k+1|} \leq 1$ for all $k \in Q_n'$. The first inequality is Chebyshev, the equality follows by stationarity together with independence, and the final inequality follows as $c_1 < 1$. Thus, $\mathbb{P}(A_n) \leq (\ln(n^p) + 1)n^{-r}$ and by Borel-Cantelli, $\mathbb{P}[A_n \ i.o.] = 0$.

□

Lemma 8. *Suppose $r > 1$. If*

$$A_n = \left\{\omega : \exists k \in [-n,n], |k-n| \leq \ln(n^p) \text{ and } \frac{1}{t_\omega(k+1)\cdots t_\omega(n)} \geq n^{\frac{r}{\beta}}\right\},$$

then $\mathbb{P}[A_n \text{ i.o.}] = 0$.

Proof. The argument is identical to the one given for Lemma 7. □

7. Proof of Theorem 10

We are now ready to give the proof of Theorem 10. We note that it essentially goes along the lines of the one in [19] with minor adjustments.

Proof. We will show $2n+1$ is $(\gamma(\tilde{E}) - 8\varepsilon_0, n, \tilde{E}, \omega)$-*regular* for all sufficiently large n. The proof that $2n$ is $(\gamma(\tilde{E}) - 8\varepsilon_0, n, \tilde{E}, \omega)$-*regular* (for sufficiently large n) is similar. Fix $0 < \epsilon_0 < \dfrac{d}{8}$ and obtain a corresponding $\eta_0 > 0$ through Corollary 2. Choose $0 < \delta_0 < \eta_0$, $0 < \varepsilon < \min\{\dfrac{\eta_0 - \delta_0}{3}, \varepsilon_0\}$, $p > \dfrac{6}{\eta_\epsilon}$, and $r > 2$. Using Lemmas 1 and 3-8, with the above $\epsilon_0, \epsilon, \delta_0$, and p and using Theorem 4, we obtain $\tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that the conclusion of Theorem 4 along with the conclusions of Lemmas 1 and 3-8 hold for all $\omega \in \tilde{\Omega}$.

Now let $\omega \in \tilde{\Omega}$ and let $\tilde{E} \in \tilde{I}$ be a generalized eigenvalue for H_ω with generalized eigenfunction ψ. We assume without loss of generality that $\psi(0) \neq 0$. Finally, we may choose N so that for $n > N$ the conclusions of Lemmas 3-8 along with the conclusion of Corollary 1 (with the above ϵ and $\tilde{E}$) hold for this ω and 0 is $(\gamma(\tilde{E}) - 8\varepsilon_0, n, \tilde{E}, \omega)$-*singular*.

Suppose that for infinitely many n ($n > N$), $2n+1$ is $(\gamma(\tilde{E}) - 8\varepsilon_0, n, \tilde{E}, \omega)$-*singular*. By Lemma 2 and Corollary 1, $\tilde{E} \in B^-_{[n+1,3n+1],\varepsilon_0,\omega}$. Note that all $2n+1$ eigenvalues of $H_{[n+1,3n+1],\omega}$ (which are all real) belong to $\tilde{B^-}_{[n+1,3n+1],\varepsilon_0,\omega}$. Since $P_{[n+1,3n+1],E,\omega}$ is a polynomial in E, it follows that $\tilde{B^-}_{[n+1,3n+1],\epsilon_0,\omega}$ is the union of at most $2n+1$ intervals around the eigenvalues of $H_{\omega,[n+1,3n+1]}$. Thus, $\tilde{E}$ is in one of these intervals. Moreover, Lemma 3 gives $m(B^-_{[n+1,3n+1],\epsilon_0,\omega}) \leq e^{-(\eta_0-\delta_0)(2n+1)}$, so since $\tilde{E} \in \tilde{I}$, we have the existence of E_j, an eigenvalue of $H_{\omega,[n+1,3n+1]}$, so that $E_j \in I$ and $|\tilde{E} - E_j| \leq e^{-(\eta_0-\delta_0)(2n+1)}$.

Applying the above argument with 0 in place of $2n+1$ yields E_i, an eigenvalue of $H_{\omega,[-n,n]}$ which lies in $B^-_{[-n,n],\varepsilon_0,\omega}$ such that $|\tilde{E} - E_i| \leq e^{-(\eta_0-\delta_0)(2n+1)}$. Thus, $|E_i - E_j| \leq 2e^{-(\eta_0-\delta_0)(2n+1)}$.
By the previous line and the fact that $E_j \notin B_{[-n,n],\epsilon,\omega}$ (by Lemma 4), $||G_{[-n,n],E_j,\omega}|| \geq \dfrac{1}{2}e^{(\eta_0-\delta_0)(2n+1)}$ and hence there exist $y_1, y_2 \in [-n,n]$, (w.log $y_1 \leq y_2$), so that

$$|G_{[-n,n],E_j,\omega}(y_1,y_2)| \geq \frac{1}{2\sqrt{2n+1}} e^{(\eta_0-\delta_0)(2n+1)}.$$

Again, using $E_j \notin B_{[-n,n],\epsilon,\omega}$, we obtain:

$$\frac{|P_{[-n,n],E_j,\omega}|}{t_\omega(-n)\cdots t_\omega(n)} \geq e^{(\gamma(E_j)-\varepsilon)(2n+1)}.$$

By recalling $|G_{[-n,n],E_j,\omega}(y_1,y_2)| = \dfrac{|P_{[-n,y-1],E_j,\omega}t_\omega(y_1)\cdots t_\omega(y_2-1)P_{[y_2+1,n],E_j,\omega}|}{|P_{[-n,n],E_j,\omega}|}$,
we have:

$$\frac{|P_{[-n,y_1-1],E_j,\omega}|t_\omega(y_1)\cdots t_\omega(y_2-1)|P_{[y_2+1,n],E_j,\omega}|}{\prod_{k=-n}^{n} t_\omega(k)} \geq \frac{e^{(\eta_0-\delta_0)(2n+1)}e^{(\gamma(E_j)-\varepsilon)(2n+1)}}{2\sqrt{2n+1}}.$$

We rewrite the left hand side of our inequality as:

$$\frac{|P_{[-n,y_1-1],E_j,\omega}|}{t_\omega(-n)\cdots t_\omega(y_1-1)}\frac{1}{t_\omega(y_2)}\frac{|P_{[y_2+1,n],E_j,\omega}|}{t_\omega(y_2+1)\cdots t_\omega(n)}. \tag{6}$$

Recall we have $y_1 \leq y_2$, so there are effectively three cases to consider:

$$|-n-y_1| \geq \ln(n^p) \text{ and } |n-y_2| \geq \ln(n^p), \tag{7}$$

$$|-n-y_1| \geq \ln(n^p) \text{ while } |n-y_2| \leq \ln(n^p), \tag{8}$$

$$|-n-y_1| \leq \ln(n^p) \text{ and } |n-y_2| \leq \ln(n^p). \tag{9}$$

For the first case, we apply Lemma 6 to the middle term in (6) and Lemma 4 to the remaining two terms to obtain:

$$n^{\frac{r}{\beta}}e^{(\gamma(E_j)+\varepsilon)(2n+1)} \geq \frac{1}{2\sqrt{2n+1}}e^{(\eta_0-\delta_0)(2n+1)}e^{(\gamma(E_j)-\varepsilon)(2n+1)}. \tag{10}$$

For the second case, we again apply Lemma 6 to the middle term in (6). We then apply Corollary 3 to the numerator of right-most term, Lemma 8 to the denominator, and Lemma 4 to the left-hand term to obtain:

$$n^{\frac{2r}{\beta}+(q(ln(n^p)+1))}e^{(\gamma(E_j)+\varepsilon)(2n+1)} \geq \frac{1}{2\sqrt{2n+1}}e^{(\eta_0-\delta_0)(2n+1)}e^{(\gamma(E_j)-\varepsilon)(2n+1)}. \tag{11}$$

And finally, for the third case, we again apply Lemma 6 to the middle term. Then, we apply Corollary 3 to the numerators of the terms on the left and the right, Lemmas 7 and 8 to the denominators to obtain:

$$n^{\frac{3r}{\beta}+2q(ln(n^p+1))} \geq \frac{1}{2\sqrt{2n+1}}e^{(\eta_0-\delta_0)(2n+1)}e^{(\gamma(E_j)-\varepsilon)(2n+1)}. \tag{12}$$

The first case leads to a contradiction by letting $n \to \infty$, since $(\gamma(E_j)-\epsilon)+(\eta_0-\delta_0) > \gamma(E_j)+\epsilon$.

For the second and third cases, the ratio of the RHS to the LHS in the above inequalities tends to ∞ as $n \to \infty$, providing the desired contradiction. We conclude that for $n > N$, $2n+1$ is $(\gamma(\tilde{E}) - 8\varepsilon_0, n, \tilde{E}, \omega)$-*regular*.
Finally, since the interval $\tilde{I}$ was arbitrary, the proof is complete.

□

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